Dispersive representation of the pion vector form factor in $\tau \to \pi\pi\nu_\tau$ decays

D. Gómez Dumm[1], P. Roig[2]

[1] IFLP, CONICET − Dpto. de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina.

[2] Grup de Física Teòrica, Institut de Física d'Altes Energies, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain.

Abstract

We propose a dispersive representation of the charged pion vector form factor that is consistent with chiral symmetry and fulfills the constraints imposed by analyticity and unitarity. Unknown parameters are fitted to the very precise data on $\tau^- \to \pi^-\pi^0\nu_\tau$ decays obtained by Belle, leading to a good description of the corresponding spectral function up to a $\pi\pi$ squared invariant mass $s \simeq 1.5$ GeV^2. We determine the $\rho(770)$ mass and width pole parameters and obtain the values of low energy observables. The significance of isospin breaking corrections is also discussed. For larger values of s, this representation is complemented with a phenomenological description to allow its implementation in the new TAUOLA hadronic currents.

1 Introduction

Last years have witnessed a notorious progress in the knowledge of the two-pion system, both from theoretical and experimental sides. In particular, the high precision measurements of the charged and neutral pion vector form factors performed at the flavour factories BaBar [1], Belle [2], CMD-2 [3], KLOE [4] and SND [5] have significantly improved the accuracy of previous data. As it is well known, theoretical predictions for these form factors cannot be obtained analytically from first principles through standard calculations, since this involves in general the hadronization of QCD currents in a nonperturbative energy regime ($E \lesssim 1$ GeV). In order to overcome this problem, one can rely on effective models [6, 7, 8] that intend to describe the involved dynamics within a simplified scheme. In general, in these models the form factors are required to satisfy the proper behaviour at very low energies, and the effect of intermediate resonances is taken into account through the inclusion of Breit-Wigner functions. In this way one can obtain phenomenologically adequate hadronic matrix elements [9, 10, 11, 12], which have been included e.g. in the standard Monte Carlo Generator for tau decays, TAUOLA [13]. However, in general these models include several ad-hoc assumptions, and can even have problems of consistency with QCD [14, 15]. In addition, even when the agreement with experimental data can be very good, usually the model parameters can be hardly related to the underlying strong interaction theory.

Another possible approach is to consider just the symmetry properties of QCD in order to build a general effective action adequate for the desired energy regime. At very low energies $E \ll M_\rho$, where M_ρ is the $\rho(770)$ resonance mass, the approximate chiral symmetry of QCD allows to build the effective quantum field theory known as Chiral Perturbation Theory (χPT) [16]. The latter provides a successful description of the low-energy phenomenology of strong and electroweak interactions, in which hadronic observables are calculated through an expansion in powers of ratios of momenta and masses of the lightest degrees of freedom (light pseudoscalar mesons) over a chiral symmetry breaking scale, $4\pi F_\pi \sim 1.2$ GeV. However, for $E \sim M_\rho$ the expansion parameters become large, and new degrees of freedom, namely the lowest-lying light-flavoured resonances, become active. Even though in this regime there is no straightforward expansion parameter, one can build an effective theory by considering an expansion in powers of the inverse of the number of colours, $1/N_C$, with the introduction of resonances as active fields in the effective action. Indeed, it is found that this approach allows to describe satisfactorily most salient features of meson phenomenology [17, 18], which suggests that the large-N_C limit of QCD is a good starting point to derive a chiral Lagrangian that includes resonance fields [19, 20, 21].

In this work we study one of the simplest hadronic observables, namely the pion vector form factor $F_V^\pi(s)$, defined through

$$\langle \pi^0 \pi^- | \bar{d} \gamma^\mu u | \emptyset \rangle \;=\; \sqrt{2} F_V^\pi(s) \, (p_{\pi^-} - p_{\pi^0})^\mu \;, \tag{1}$$

where $s \equiv q^2 \equiv (p_{\pi^-} + p_{\pi^0})^2$. For $s > 0$, this form factor is probed by the decay $\tau^- \to \pi^- \pi^0 \nu_\tau$, while in the isospin symmetry limit it can be experimentally measured from $e^+ e^- \to \pi^+ \pi^-$ ($s > 0$) and elastic $e^- \pi^+$ scattering ($s < 0$). The analysis of $F_V^\pi(s)$ allows to increase our knowledge of the hadronization of

QCD currents in the intermediate energy region, where the presence of meson resonances plays a crucial role. On the other hand, the analysis of isospin breaking corrections to the form factors in tau decays and e^+e^- scattering [22, 23, 24, 25] is essential for the evaluation of the hadronic contribution to the anomalous magnetic moment of the muon a_μ, which provides a stringent test of new physics [26].

The theoretical analysis of the pion vector form factor has been addressed by several authors in the last years. At very low energies, $F_V^\pi(s)$ has been computed in χPT up to $\mathcal{O}(p^6)$ [27, 28, 29]. Then, to enlarge the domain of applicability up to ~ 1 GeV, unitarization techniques [30, 31] and dispersion relations have been employed [32, 33, 34]. Moreover, in order to go beyond this energy region, the inclusion of the $\rho(1450)$ resonance [35] and even a tower of resonances, inspired in the $N_C \to \infty$ limit [36, 37], have been proposed. Our work is a sort of extension of those in Refs. [32, 33, 38, 39], in which the authors analyze $\pi\pi$ and $K\pi$ vector form factors considering $\mathcal{O}(p^4)$ expressions obtained from a chiral effective theory that includes the dominant resonance exchange, followed by an Omnès-like resummation of final state interactions. Our procedure is similar to that proposed in Ref. [39] for the $K\pi$ vector form factor: we consider an n-subtracted dispersion relation for $F_V^\pi(s)$ in which the input elastic phase shift $\delta_1^1(s)$ is taken from the effective theory, resumming the chiral loops into the denominator of the $\mathcal{O}(p^4)$ form factor. This ensures to fulfill unitarity and analyticity constraints. It is seen that a phenomenologically good result is obtained with three subtractions, hence our expression for $F_V^\pi(s)$ depends on four parameters: M_ρ, F_π, and two subtraction constants α_1 and α_2 (the remaining subtraction constant is fixed by the normalization of the form factor). These constants can be related to chiral low-energy observables [28, 29], namely the squared charged pion radius $\langle r^2 \rangle_V^\pi$ and the coefficients of $\mathcal{O}(s^2)$ and $\mathcal{O}(s^3)$ terms in the chiral expansion, c_V^π and d_V^π, respectively.

The above described approach is able to provide a good description of the $\tau^- \to \pi^-\pi^0\nu_\tau$ spectral function for a squared $\pi\pi$ invariant mass up to about $s_{\max} \simeq 1.5$ GeV^2. Beyond these energies, we propose a complementary expression for the form factor that includes the effects of the excited states ρ' and ρ'', matching smoothly the previous one at $s \sim s_{\max}$. Our result for the full form factor can be useful to improve the new version of TAUOLA [40], which presently includes the expressions obtained within a chiral Lagrangian framework with resonances [41, 42, 43, 44, 45]. This is important [46] not only for the proper simulation of backgrounds and subsequent signal extraction at the more frequent tau decay modes but also for the analysis of rare processes and the searches of new physics [47].

The article is organized as follows: in Section 2 we obtain a dispersive representation of $F_V^\pi(s)$ and discuss the inclusion of isospin breaking corrections. The model parameters are fitted to experimental data up to $s \simeq 1.5$ GeV^2, and our input for $\delta_1^1(s)$ in the elastic region is confronted with present experimental values. Then, in Section 3 we extend our parametrization of $F_V^\pi(s)$ to higher energies, including the effective contribution of excited resonances. The agreement with experimental data is shown. In Section 4 we present the results for the low-energy observables related to our subtraction constants. Finally, in Section 5 we state the conclusions of our analysis.

2 Low energy description of $F_V^\pi(s)$

As stated, at very low energies the pion vector form factor is well described by χPT. Let us first consider the limit of exact isospin symmetry. At $\mathcal{O}(p^4)$, one has [48]

$$F_V^\pi(s)^{\chi\text{PT}} \;=\; 1 \,+\, \frac{2L_9^r(\mu)}{F_\pi^2}\, s \,-\, \frac{s}{96\pi^2 F_\pi^2}\left[A_\pi(s,\mu^2) + \frac{1}{2}A_K(s,\mu^2)\right] \;, \qquad (2)$$

where $L_9^r(\mu)$ is one of the renormalized low-energy coupling constants in the chiral Lagrangian. The functions $A_P(s,\mu^2)$ are given by

$$A_P(s,\mu^2) \;=\; \log\frac{m_P^2}{\mu^2} \,+\, 8\frac{m_P^2}{s} \,-\, \frac{5}{3} \,+\, \sigma_P^3(s)\,\log\left(\frac{\sigma_P(s)+1}{\sigma_P(s)-1}\right) \;, \qquad (3)$$

where the phase space function $\sigma_P(s)$ reads

$$\sigma_P(s) = \sqrt{1 - 4\frac{m_P^2}{s}} \;. \qquad (4)$$

On the other hand, the computation of $F_V^\pi(s)$ from a chiral Lagrangian that includes the lowest-lying vector meson multiplet as active resonance fields yields [19]

$$F_V^\pi(s) \;=\; 1 \,+\, \frac{F_V\,G_V}{F_\pi^2}\,\frac{s}{M_V^2 - s} \;, \qquad (5)$$

where $M_V = M_\rho$, and F_V and G_V measure the strength of the ρV_μ and $\rho\pi\pi$ couplings, respectively, V_μ being the quark vector current. This tree level result corresponds to the leading term in powers of $1/N_C$, and it is $\mathcal{O}(p^4)$ in the chiral expansion. If the form factor is required to vanish in the limit of large s, then one gets the relation $F_V G_V = F_\pi^2$, which yields

$$F_V^\pi(s) = \frac{M_\rho^2}{M_\rho^2 - s} \;. \qquad (6)$$

Comparing with Eq. (2), the low energy χPT coupling L_9 is predicted to be

$$L_9 = \frac{F_V G_V}{2M_\rho^2} = \frac{F_\pi^2}{2M_\rho^2} \simeq 7.2\cdot 10^{-3} \;, \qquad (7)$$

in very good agreement with the value obtained from phenomenology. This shows explicitly that the $\rho(770)$ contribution is the dominant physical effect in the vector form factor of the pion. Now, as stated in Ref. [32], one can do better and match Eq. (6) to the $\mathcal{O}(p^4)$ χPT result in Eq. (2), including the final state interactions encoded in the chiral loop functions $A_P(s,\mu^2)$:

$$F_V^\pi(s) = \frac{M_\rho^2}{M_\rho^2 - s} - \frac{s}{96\pi^2 F^2}\left[A_\pi(s) + \frac{1}{2}A_K(s)\right] \;. \qquad (8)$$

We omit from now on the explicit dependence on the μ scale, taking $\mu = M_\rho$. The results do not depend significantly on changes in this scale. From Eq. (8),

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unitarity and analyticity constraints lead to the Omnès exponentiation of the full loop function [32],

$$F_V^\pi(s) = \frac{M_\rho^2}{M_\rho^2 - s} \exp\left\{ -\frac{s}{96\pi^2 F^2} \left[A_\pi(s) + \frac{1}{2} A_K(s) \right] \right\} . \tag{9}$$

In order to account for the resonance width, here one should not simply replace $M^2 - s$ by $M^2 - s - iM\Gamma(s)$ in the effective propagator, since this would double count $\Im m[A_P(s)]$ and analyticity would be violated at $\mathcal{O}(p^6)$ in the chiral expansion. One could avoid the double counting by shifting the imaginary part of the loop functions from the exponential to the propagator [32], but still analyticity would be lost. We follow instead a procedure similar to that proposed in Ref. [39] for the $K\pi$ form factor, in which unitarity and analyticity are preserved. As a starting point we consider a form factor in which the loop functions are resummed into the denominator,

$$
\begin{aligned}
F_V^{\pi\,(0)}(s) &= \frac{M_\rho^2}{M_\rho^2 \left[1 + \frac{s}{96\pi^2 F_\pi^2} \left(A_\pi(s) + \frac{1}{2} A_K(s) \right) \right] - s} \\
&= \frac{M_\rho^2}{M_\rho^2 \left[1 + \frac{s}{96\pi^2 F_\pi^2} \Re e \left(A_\pi(s) + \frac{1}{2} A_K(s) \right) \right] - s - i M_\rho \Gamma_\rho(s)} , \tag{10}
\end{aligned}
$$

where we have defined the imaginary part of the denominator as $-M_\rho \Gamma_\rho(s)$. The energy dependent width is thus given by

$$\Gamma_\rho(s) = -\frac{M_\rho\, s}{96\,\pi^2\, F_\pi^2} \Im m\left[A_\pi(s) + \frac{1}{2} A_K(s) \right] , \tag{11}$$

and from Eq. (3) one has

$$\Gamma_\rho(s) = \frac{s\, M_\rho}{96\,\pi\, F_\pi^2} \left[\theta\left(s - 4m_\pi^2 \right) \sigma_\pi^3(s) + \frac{1}{2} \theta\left(s - 4m_K^2 \right) \sigma_K^3(s) \right] , \tag{12}$$

which is in agreement with the result obtained from a chiral theory with resonances [49] if one assumes the relation $G_V = F_V/2$.

The form factor in Eq. (10) has the correct low-energy behaviour at $\mathcal{O}(p^4)$ [32] and leading $\mathcal{O}(p^6)$ contributions in χPT [50], and vanishes at short distances as expected from the asymptotic behaviour ruled by QCD. As stated, the loop functions $A_P(s)$ contain the logarithmic corrections induced by final state interactions. Now we take into account the fact that the two pion vector form factor is an analytic function in the complex plane, except for a cut along the positive real axis starting at the threshold for two pion production, $s_{\text{thr}} = 4m_\pi^2$, where its imaginary part develops a discontinuity. From unitarity it can be shown [32, 33] that the form factor satisfies an n-subtracted dispersion relation that involves the scattering phase in the elastic region, for which experimental data are available. In the case of n subtractions at $s = 0$, the dispersion relation admits the well-known Omnès solution

$$F_V^\pi(s) = P_n(s) \exp\left\{ \frac{s^n}{\pi} \int\limits_{s_{\text{thr}}}^{\infty} ds' \frac{\delta_1^1(s')}{(s')^n(s' - s - i\epsilon)} \right\} , \tag{13}$$

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where

$$\log P_n(s) \;=\; \sum_{k=0}^{n-1} \alpha_k \, \frac{s^k}{k!} \tag{14}$$

is the corresponding subtraction polynomial, and $\delta_1^1(s)$ is the $I = 1$, $J = 1$ two-pseudoscalar scattering phase shift. The subtraction constants α_k are given by

$$\alpha_k \;=\; \frac{d^k}{ds^k} \log F_V^\pi(s) \bigg|_{s=0} . \tag{15}$$

At least one subtraction is required in Eq. (13) to achieve convergence. Here the first subtraction constant has been fixed from the normalization $F_V^\pi(0) = 1$, which holds with good approximation in view of the conservation of the vector current in the isospin symmetry limit. On the other hand, in order to determine the scattering phase shift $\delta_1^1(s)$ [to be used as input in Eq. (13)] we follow the approach in Ref. [39], taking

$$\tan \delta_1^1(s) = \frac{\Im m F_V^{\pi(0)}(s)}{\Re e F_V^{\pi(0)}(s)} \;, \tag{16}$$

where $F_V^{\pi(0)}(s)$ is given by Eq. (10). In this way, the form factor in Eq. (10) trivially satisfies the Omnés relation (13) for $n = 1$ and $F_V^\pi(0) = 1$. This form factor should be adequate to reproduce the experimental observations in the low energy limit, since by construction it matches χPT results. However, beyond this limit one would not expect a sufficiently accurate description of the data. Fortunately, the analyticity properties of $F_V^\pi(s)$ allow to increase accuracy by considering more subtractions in Eq. (13): for higher n, the weight of the dispersive integral at large energies gets reduced, and the corresponding information is translated to the subtraction constants [33], which can be taken as unknown parameters. In addition, some approach has to be used to deal with the phase shift beyond the inelastic two-kaon threshold, where the contribution of the dispersive integral is in general still relevant and Eq. (13) is no longer valid (in fact, this happens already at the four-pion threshold, but higher multiplicity intermediate states are expected to be phase space suppressed). The goal is to obtain a form factor that leads to a satisfactory description of the available data, considering just a few subtractions and a phenomenologically adequate elastic phase shift.

On the basis of the previous discussion, we have carried out fits of $F_V^\pi(s)$ to Belle data from τ decays. We find that a good description of the data can be obtained with $n = 3$ subtractions, i.e. taking

$$F_V^\pi(s) \;=\; \exp\left[\alpha_1 s \,+\, \frac{\alpha_2}{2}\, s^2 \,+\, \frac{s^3}{\pi} \int_{s_{\mathrm{thr}}}^\infty ds' \, \frac{\delta_1^1(s')}{(s')^3(s' - s - i\epsilon)} \right] . \tag{17}$$

In this form factor we have four parameters, namely the subtraction constants $\alpha_{1,2}$, and the parameters M_ρ and F_π that determine the phase shift δ_1^1 according to Eqs. (10) and (16). In addition, in order to deal with the phase shift in the large energy region, we have distinguished two intermediate (squared) energies s_1 and s_2. The former is defined as the limit up to which we consider Eq. (16) to be a reliable description of the phase shift. As stated, we expect this value to

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be of the order of the inelastic two-kaon threshold, $s_1 \simeq 4M_K^2$, or alternatively we can consider the limit $s_1 \simeq (M_\rho + \Gamma_\rho)^2$, where the effect of the ρ resonance should dominate. In any case one expects s_1 to be about 1 GeV^2. As we show below, this will be supported *a posteriori* by the good agreement between our predictions and the experimental data for δ_1^1 quoted some time ago in Refs. [51, 52, 53]. Beyond s_1, the dispersive integral in Eq. (17) should be affected not only by inelastic contributions but also by the presence of excited resonance states. The other point, s_2, indicates the energy at which we assume that the phase shift saturates its asymptotic value $\delta_1^1(s \to \infty) = \pi$, corresponding to the existence of a single narrow resonance [54, 55, 56][1]. Here we take $s_2 \simeq M_\tau^2$. In the intermediate region, $s_1 \leq s \leq s_2$, we assume for simplicity a linear behaviour of δ_1^1 with s. In order to take into account the uncertainties arising from these assumptions, when performing our fits we have considered possible variations of the values of s_1 and s_2, and of the upper integration limit s_∞, which is usually taken to be in the range $[2.25, \infty]$ GeV^2 [33, 39, 57]. The corresponding effects on our results have been taken as part of the systematic error of our theoretical approach.

Another aspect to be taken into account is the effect of isospin violating corrections to the pion vector form factor. In general one has to distinguish between the neutral and charged pion vector form factors, the latter being defined by Eq. (1). The corrections can be expanded in powers of the quark mass difference and the electromagnetic coupling, in addition to the chiral counting. At the leading order, the spectral function for the decay $\tau^- \to \pi^- \pi^0 \nu_\tau$ can be written as [58]

$$\frac{d\Gamma(\tau^- \to \pi^- \pi^0 \nu_\tau)}{ds} = \frac{G_F^2 \, m_\tau^3}{384 \, \pi^3} \, S_{EW} \, |V_{ud}|^2 \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right)$$
$$\lambda^{3/2}\left(1, \frac{m_{\pi^0}^2}{s}, \frac{m_{\pi^+}^2}{s}\right) |f_+(s)|^2 \, G_{EM}(s) , \qquad (18)$$

where $S_{EW} = 1 + (\alpha/\pi)\log(M_Z^2/M_\tau^2)$ includes the dominant short-distance electroweak corrections, and the factor $G_{EM}(s)$ arises from the contribution of electromagnetic loops. In the isospin limit one has $S_{EW} = G_{EM} = 1$, $\lambda^{1/2}(1, m_\pi^2/s, m_\pi^2/s) = \sigma_\pi(s)$, and the form factor in Eq. (18) reduces to the pion vector form factor, $f_+(s) = F_V^\pi(s)$. The dominant isospin breaking effect in $F_V^\pi(s)$ is that arising from phase space, i.e., from considering different masses for the charged and neutral pions and kaons in the loop functions. Thus one has to replace the functions $A_\pi(s)$ and $A_K(s)$ by $A_{\pi^-\pi^0}(s)$ and $A_{K^-K^0}(s)$, respectively. Explicit expressions for these functions are given in Appendix A. In this way, following the same steps that lead to Eq. (17) we can obtain a dispersion relation for the charged pion vector form factor, $F_V^{\pi^+}(s)$. In addition, in $f_+(s)$ one has to take into account a local electromagnetic correction f_{local}^{elm}, which contributes as an additional term in the decay amplitude [58]. One has then

$$f_+(s) = F_V^{\pi^+}(s) + f_{local}^{elm} . \qquad (19)$$

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[1]The asymptotic limit of the phase shift obtained from Eq. (16) slightly deviates from this value, owing to the linear growth of Γ_ρ with s.

This local electromagnetic correction is given by [58]

$$f_{\text{local}}^{\text{elm}} = \frac{\alpha}{4\pi} \left(-\frac{3}{2} - \frac{1}{2} \log \frac{M_\tau^2}{\mu^2} - \log \frac{m_\pi^2}{\mu^2} + 2 \log \frac{M_\tau^2}{M_\rho^2} - X(\mu) \right) , \qquad (20)$$

where the scale dependence in the last term cancels that in the logarithms. At the scale $\mu = M_\rho$, $X(\mu)$ is estimated to be between -2.5 and 4.5 [58]. Finally, the loop correction $G_{EM}(s)$ has been computed (including resonance contributions) in Refs. [59] and [60]. We notice that this correction has not been taken into account in the extraction of the form factor carried out by the Belle Collaboration in Refs. [2, 61] (it has been included for the analysis of the muon anomalous magnetic moment, where isospin breaking effects represent a central subject of interest).

In order to incorporate the effect of isospin corrections and evaluate its significance, we have carried out our fits for the tau decay data considering three different situations:

- I) The limit of exact isospin symmetry, in which $f_+(s) = F_V^\pi(s)$, where the form factor is given by Eq. (17).

- II) The inclusion of isospin breaking corrections at the level of kinematics, i.e. considering different masses for the charged and neutral particles in the loop functions and the kinematical factors in Eq. (18) (this would correspond to Belle's analysis [2] of the $\tau^- \to \pi^- \pi^0 \nu_\tau$ decay width).

- III) The inclusion of all lowest order isospin breaking corrections, as in Eqs. (18-20). For the factor $G_{EM}(s)$ we have considered here the analysis in Ref. [60].

As a general result, it is found that we obtain a good fit to Belle data [2] for $s \lesssim 1.5$ GeV^2. Our fits have been carried out with the MINUIT package taking the first 30 points ($s_{\text{max}} = 1.525$ GeV^2, with a bin width of 0.05 GeV^2). The results are shown in Table 1. First of all, it is worth to notice that —even with just a few input parameters— the theoretical curve is able to fit the very precise set of experimental data with a χ^2/dof value close to unity. Hence, it is seen that within this energy range the data can be described without the inclusion of higher resonant states in the theoretical scheme. On the other hand, it is found that the effect of isospin breaking corrections on the fitted parameters is below the 2% level.

Regarding the errors in the fitted parameters, we have quoted separately those arising from the fit and the systematic errors coming from the theoretical approach. The latter are basically due to the uncertainties in the energy range to be fitted, the number of subtractions considered, and the values of s_1, s_2 and s_∞ in the dispersive integral. In order to have an estimation of the effect of these uncertainties we have considered the fits for s_{max} in the range $[1.325, 1.525]$ GeV^2, 2 to 4 subtractions, and s_1, s_2, s_∞ in the ranges $[0.95, 1.1]^2$, $[M_\tau^2, \infty]$ and $[2.25, \infty]$ GeV^2, respectively. The corresponding results have been quoted in the second brackets in Table 1, while the numbers in the first brackets stand for the statistical errors arising from the fit. It is found that M_ρ and F_π appear to be anticorrelated, and the same happens with the parameters α_1 and α_2.

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[2]Best fits are obtained in all cases for $s_1 \simeq 0.98$ GeV^2, in agreement with theoretical expectations.

	Fit value (I)	Fit value (II)	Fit value (III)
$M_\rho\,[\text{GeV}]$	0.8430(5)(17)	0.8427(5)(14)	0.8426(5)(20)
$F_\pi\,[\text{GeV}]$	0.0901(2)(5)	0.0902(2)(4)	0.0906(2)(4)
$\alpha_1\,[\text{GeV}^{-2}]$	1.87(1)(3)	1.87(1)(3)	1.81(1)(2)
$\alpha_2\,[\text{GeV}^{-4}]$	4.29(1)(7)	4.31(1)(7)	4.40(1)(6)
χ^2/dof	1.37	1.37	1.55
$\Gamma_\rho(M_\rho^2)\,[\text{GeV}]$	0.206(1)(3)	0.206(1)(3)	0.204(1)(3)

Table 1: Results of our fits. The first and second numbers in brackets correspond to the statistic and theoretical systematic errors, respectively. $\Gamma_\rho(M_\rho^2)$ is obtained using the fitted values of M_ρ and F_π and is given only for reference.

From the table it is seen that the central values of F_π obtained from the fit are about two percent below the value of 92.2 MeV quoted by the PDG [62]. The difference can be attributed to further theoretical uncertainties, mainly arising from the effect of higher order terms in the large N_C expansion. This includes corrections to the relations $G_V = F_V/2$ and $G_V F_V = F_\pi^2$, which have been used for the matching between the form factors obtained within the low energy χPT theory and the chiral theory with resonances. In fact, as already pointed out in Refs. [33, 39], the energy-dependent width given by the imaginary part of the loop function with $F_\pi = 92.2$ MeV is not adequately normalized so as to reproduce both the experimental data on $\pi\pi$ and $K\pi$ tau decay channels. Concerning the properties of the $\rho(770)$ resonance, in order to obtain the corresponding physical mass and width one should compute the position of the pole of the pion vector form factor in the complex s plane, say s_{pole}. One has

$$\sqrt{s_{\text{pole}}} = M_\rho^{\text{pole}} - \frac{i}{2}\Gamma_\rho^{\text{pole}}\ . \tag{21}$$

Unfortunately, s_{pole} cannot be obtained directly from the expression for the pion vector form factor in Eq. (17), since in general the complex variable s in the dispersion relation is not in the same Riemann sheet in which the pole is located. In order to deal with this difficulty, one possible procedure is to make use of one-pole Padé approximants $P_1^N(s; s_0)$, defined by

$$P_1^N(s; s_0) = \sum_{k=0}^{N-1} a_K(s - s_0)^k + \frac{a_N(s - s_0)^N}{1 - \frac{a_{N+1}}{a_N}(s - s_0)}\ . \tag{22}$$

In general, if one assumes that a complex function $F(s)$ is analytical in a disk around some point s_0 except at a point s_{pole}, where it has a single pole, then the de Montessus de Ballore's theorem [63] states that the sequence of one-pole Padé approximants $P_1^N(s; s_0)$ converges to $F(s)$ in any compact subset of the disk excluding the pole. Hence, the Padé pole $z_p = s_0 + a_N/a_{N+1}$ of $P_1^N(s; s_0)$ converges to s_{pole} for $N \to \infty$. The application of this method for the analysis of resonance poles has been previously considered in Refs. [64], where details can be found. In our case we have approximated the form factor $F_V^\pi(s)$ with a function of the type of that in Eq. (22), taking $s_0 = (M_0 - i\Gamma_0/2)^2$, with $M_0 = 0.77$ GeV, $\Gamma_0 = 0.15$ GeV. The coefficients a_K, $K = 1, \ldots, N+1$ ($a_0 = 1$ owing to vector current conservation) have been determined from a fit to a

set of values of $|F_V^\pi(s)|$ and $\delta_1^1(s)$ obtained from our dispersive representation, Eq. (17), between the first and second production thresholds. From the results of this fit, taking Padé approximants with $N = 5$ and $N = 6$ coefficients, we find

$$M_\rho^{\text{pole}} = (759 \pm 2)\,\text{MeV}\,, \quad \Gamma_\rho^{\text{pole}} = (146 \pm 6)\,\text{MeV} \quad (\text{Fit I})\,;$$
$$M_\rho^{\text{pole}} = (760 \pm 2)\,\text{MeV}\,, \quad \Gamma_\rho^{\text{pole}} = (147 \pm 6)\,\text{MeV} \quad (\text{Fit III})\,. \quad (23)$$

This turns out to be our best determination of s_{pole}. One can still increase N and get a better fit of the data set, but given the larger number of parameters, the errors of a_N and a_{N+1} become also larger. As expected, the values in Eq. (23) are not modified either if we take a different input for s_0 or if we increase the number of values of $|F_V^\pi(s)|$ and $\delta_1^1(s)$ to be fitted.

For comparison, we have also analyzed the results for the pole mass and width of the ρ meson corresponding to the parametrization proposed time ago by Gounaris and Sakurai (GS) [6], which has been used in the fits carried out by the Belle Collaboration [65][3]. The results obtained by Belle using the normalization $F_V^\pi(0) = 1$ yield the parameter values $M_\rho^{(\text{GS})} = (774.6 \pm 0.5)$ MeV, $\Gamma_\rho^{(\text{GS})} = (148.1 \pm 1.7)$ MeV. Taking into account the prescriptions in Refs. [66, 67, 68] to deal with the cuts in the complex functions entering the GS form factor [see Eqs. (31, 32) below] these parameters correspond to

$$M_\rho^{\text{pole}} = (760.9 \pm 0.6)\,\text{MeV}\,, \quad \Gamma_\rho^{\text{pole}} = (142.2 \pm 1.6)\,\text{MeV}\,. \quad (24)$$

The results in Eqs. (23) and (24) are consistent with each other and somewhat different from the average values for the ρ mass and width quoted by the PDG, namely $M_\rho = 775.49 \pm 0.34$ MeV and $\Gamma_\rho = 149.1 \pm 0.8$ MeV [62]. In fact, the PDG values correspond to the parameters appearing in phenomenological amplitudes where the resonances are introduced through BW functions (as e.g. $M_\rho^{(\text{GS})}$ and $\Gamma_\rho^{(\text{GS})}$), hence they are strongly model dependent[4]. Alternatively, one can take the pole mass and width as the relevant resonance properties. We consider the agreement between the results in Eqs. (23) and (24) as a check of consistency, in the sense that one expects the pole parameters to be essentially model independent. For comparison, in Table 2 we show other determinations of the pole mass and width quoted in the literature. The results for M_ρ^{pole} and $\Gamma_\rho^{\text{pole}}$ obtained either from our dispersive approach or from the simple GS parametrization are found to be in good agreement with the average value of these determinations. It is seen that the errors in Eqs. (24) are smaller than those in our results, since the former were obtained from a direct fit to experimental data, while in our determination one has an additional uncertainty introduced by the Padé approximants. However, the GS parametrization represents just a simple ad-hoc description of the underlying dynamics, thus it implicitly includes a theoretical systematic error which is hardly estimable.

As a further check of consistency, we can compare the phase of the form factor obtained from our fit with present experimental data on $\delta_1^1(s)$ from Ochs et al. [51], Estabrooks and Martin [52] and Protopopescu et al. [53]. The results are shown in Fig. 1. We quote the data from threshold up to $s_1 \simeq 1\,\text{GeV}^2$, i.e. the

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[3]Details of this parametrization are given in the next section, see Eq. (31) and below.

[4]This has been pointed out in Refs. [39, 57] for the case of the $K^\star(892)$ resonance, analyzed in the context of $\tau^- \to (K\pi)^- \nu_\tau$ decays.

Reference	M_ρ^{pole}	$\Gamma_\rho^{\mathrm{pole}}$	Data	Analysis
Sanz-Cillero *et al.* [35]	$764.1^{+4.8}_{-3.7}$	$148.2^{+2.5}_{-6.2}$	$\tau \,\&\, e^+e^-$	DSE
Ananthanarayan *et al.* [69]	762.5 ± 2	142 ± 7	$\pi\pi \to \pi\pi$	RE
Feuillat *et al.* [70]	758.3 ± 5.4	145.1 ± 6.3	$\tau \,\&\, e^+e^-$	SMA
Peláez [71]	754 ± 18	148 ± 20	$\pi\pi \to \pi\pi$	UχPT
Zhou *et al.* [72]	763.0 ± 0.2	139.0 ± 0.5	$\pi\pi \to \pi\pi$	χU
Masjuan *et al.* [64]	763.7 ± 1.2	144 ± 3	τ	RA
Results from our fit I	759 ± 2	146 ± 6	τ	DR
Results from our fit III	760 ± 2	147 ± 6	τ	DR
Results from GS model	760.9 ± 0.6	142.2 ± 1.6	τ	GS

Table 2: Comparison between different results for the pole mass and width of the $\rho(770)$ meson (values are in MeV). Abbreviations for the type of analysis carried out are DSE: Dyson-Schwinger equations; RE: Roy equations; SMA: S matrix approach; UχPT: Unitarized Chiral Perturbation Theory; χU: Chiral unitarization; RA: Rational approximants; DR: Dispersive representation; GS: Gounaris-Sakurai parametrization.

region in which the phase of $F_V^{\pi\,(0)}(s)$ has been used as input for the dispersive integral. In general it is seen that the agreement is very good. It is remarkable, however, that our predictions are somewhat below the data in the region of very low energies. On the other hand, as shown in the figure, our results in that region are in very good agreement with those recently obtained in Ref. [73] using once-subtracted Roy-like equations [74]. Within errors, we also find agreement with the values obtained supplementing Roy equations with chiral symmetry constraints [75, 76]. This discrepancy between theory and experiment in the very low energy region would deserve further tests, taking into account that the experimental information corresponds to rather old measurements.

Another result that can be derived from our analysis is the value of the point $s_{\pi/2}$, satisfying $\delta_1^1(s_{\pi/2}) = \pi/2$. This point is often used for the definition of the so-called peak mass or visible mass (since it can be directly extracted from experimental data). According to our fits, this value is rather stable with respect to changes in the above mentioned systematic sources of theoretical errors. We find

$$\sqrt{s_{\pi/2}} = (775.0 \pm 0.2)\,\mathrm{MeV} \ . \tag{25}$$

This is in agreement with the result of Ref. [69], $\sqrt{s_{\pi/2}} = (774 \pm 3)$ MeV, within a Roy equation analysis of the different partial waves in $\pi\pi$ scattering. The peak mass has to coincide with the pole mass [68] when the corresponding resonance is sufficiently narrow, it lies far from other resonances, and there is a negligible background (non-resonant) contribution.

Finally, we point out that our results for the subtraction constants α_1 and α_2 are compatible with previous determinations, although the errors are smaller owing to the high quality of present Belle data. We come back to this issue in Section 4.

3 Intermediate energy description of $F_V^\pi(s)$

The approach presented in Sect. 2 has been used to obtain a theoretical description of the $\tau \to \pi\pi\nu_\tau$ spectral function for $\pi\pi$ invariant masses up to

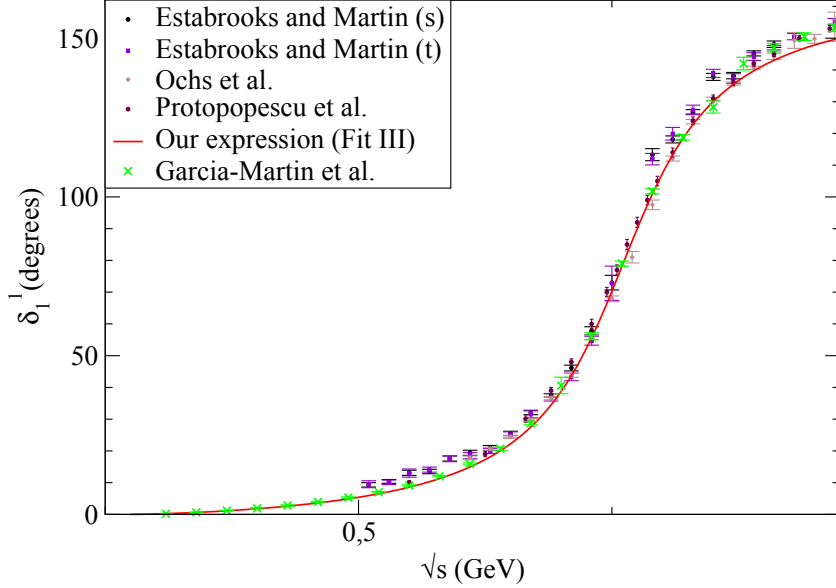Figure 1: Two-pion phase shift δ_1^1 as function of the $\pi\pi$ invariant mass squared. Our theoretical expression (red curve) is shown to be in good agreement with experimental data (from Ochs *et al.* [51], Estabrooks and Martin [52] in the s and t channels and Protopopescu *et al.* [53]) up to the opening of the two-kaon threshold, $s_1 \simeq 1$ GeV^2. At very low energies, where no data are available, our prediction agrees with the results of García-Martín *et al.* [73].

$s_{\max} \simeq 1.5$ GeV^2. Above these energies this description is not adequate, in fact, the experimental data are compatible with the presence of excited resonances. In order to complement the dispersive representation for the pion vector form factor proposed in the previous Section, we propose for the energy region $s_{\max} \lesssim s \leq M_\tau^2$ an effective form factor that includes two excited states, namely the $\rho'(1465)$ and $\rho''(1700)$ resonances. We stress that the dispersive representation is insensitive to the modelling of this "intermediate" energy region. The dynamics related with the excited resonance multiplets is poorly known, therefore there are no grounds to go beyond a phenomenological parametrization that includes several unknown parameters. We extend the form factor in Eq. (10) as [77]

$$
\begin{aligned}
F_V^\pi(s) &= \frac{M_\rho^2 + (\alpha' e^{i\phi'} + \alpha'' e^{i\phi''}) s}{M_\rho^2 \left[1 + \frac{s}{96\pi^2 F_\pi^2} \left(A_\pi(s) + \frac{1}{2} A_K(s)\right)\right] - s} \\
&\quad - \frac{\alpha' e^{i\phi'} s}{M_{\rho'}^2 \left[1 + s\, C_{\rho'} A_\pi(s)\right] - s} - \frac{\alpha'' e^{i\phi''} s}{M_{\rho''}^2 \left[1 + s\, C_{\rho''} A_\pi(s)\right] - s} \;,\quad (26)
\end{aligned}
$$

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where the constants $C_{\rho'}$ and $C_{\rho''}$ are given by

$$C_R = \frac{\Gamma_R}{\pi \, M_R^3 \, \sigma_\pi^3(M_R^2)} \; . \qquad (27)$$

The resonance masses $M_{\rho',\rho''}$ and on-shell widths $\Gamma_{\rho',\rho''}$ are free parameters of this effective form factor. By construction, the off-shell widths of the excited resonances behave in a similar way as the ρ width [44], considering the two-pion states as the dominant absorptive parts of the corresponding self-energies:

$$\Gamma_R(s) = \Gamma_R \, \frac{s}{M_R^2} \, \frac{\sigma_\pi^3(s)}{\sigma_\pi^3(M_R^2)} \, \theta(s - 4m_\pi^2) \; . \qquad (28)$$

In addition, the form factor includes the coefficients α' and α'', which measure the relative weight between the contributions of different resonances, and the phases ϕ' and ϕ'', which account for the corresponding interference.

Now the unknown parameters can be fitted to Belle data on the $\tau \to \pi\pi\nu_\tau$ spectral function. The quality of the matching between the phenomenological form factor in Eq. (26) and the dispersive representation in Eq. (10) can serve as a test of the consistency of our approach. The results of our fit for the resonance parameters can be translated to the corresponding pole values, leading to

$$M_{\rho'}^{\text{pole}} = (1.44 \pm 0.08) \text{ GeV} , \quad \Gamma_{\rho'}^{\text{pole}} = (0.32 \pm 0.08) \text{ GeV} ,$$

$$M_{\rho''}^{\text{pole}} = (1.72 \pm 0.09) \text{ GeV} , \quad \Gamma_{\rho''}^{\text{pole}} = (0.18 \pm 0.09) \text{ GeV} , \qquad (29)$$

in good agreement with the values quoted by the PDG [62]. For the coefficients and phases we obtain

$$\begin{aligned} \alpha' &= 0.08^{+0.03}_{-0.01} & \phi' &= 0.14^{+0.10}_{-0.08} \\ \alpha'' &= 0.03 \pm 0.01 & \phi'' &= 3.14^{+0.50}_{-0.06} \; . \end{aligned} \qquad (30)$$

Here, besides the statistical errors, we have included a systematic error arising from the election of the initial value of the considered energy range, say s_0. We have taken $s_0 \in [1.3, 1.55]$ GeV^2, and considered fit results with $\chi^2/dof \leq 1$. Within this range we obtain a good matching to the form factor in Eq. (10) at $s \simeq 1.35$ GeV^2. The fits are not significantly sensitive to the ρ meson parameters, which have been taken from the results in Table 1. Our final curve for the pion vector form factor covering the full range of values from threshold to M_τ^2 is shown in Fig. 2 (solid line). The quality of the fits is reflected in the good agreement between our results and the experimental data obtained by Belle, in particular in the low energy region, where the latter are very precise. In addition, it can be seen that the matching at $s \simeq 1.35$ GeV^2 is smooth, which supports the consistency of the phenomenological description proposed for the intermediate energy region. In order to appreciate the agreement with data with more detail, two close-ups of Fig. 2, corresponding to the low energy and the peak regions, are shown in Fig. 3.

It is worth to point out that the phenomenological form factor in Eq. (26) is qualitatively similar to the GS parametrization [6] mentioned in the previous section. Indeed, the GS form factor is built as a sum of Breit-Wigner-like functions that keep a nontrivial real contribution in the corresponding denominators:

$$F_V^{\pi(\text{GS})}(s) = \frac{1}{1 + \beta + \gamma} \left[BW_\rho^{\text{GS}}(s) + \beta \, BW_{\rho'}^{\text{GS}}(s) + \gamma \, BW_{\rho''}^{\text{GS}}(s) \right] , \qquad (31)$$
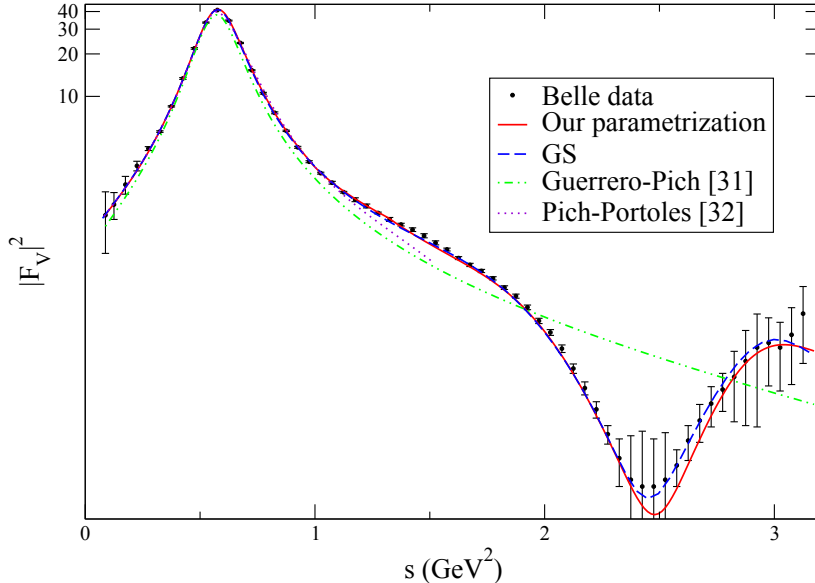
Figure 2: Pion vector form factor $F_V^\pi(s)$ compared to Belle data [2] (black dots). Solid and dashed lines correspond to our description and the GS parametrization, respectively. The dashed-dotted curve stands for the result from Ref. [32] (for $M_\rho = 775$ MeV), while the dotted line corresponds to the dispersive representation in Ref. [33] (for $\alpha_1 = 1.83$ GeV^{-2}, $\alpha_2 = 4.32$ GeV^{-4} and $M_\rho = 774.2$ MeV).

where

$$BW_R^{\mathrm{GS}}(s) = \frac{M_R^2 \left(1 + d_R \Gamma_R(s)/\sqrt{s}\right)}{(M_R^2 - s) + f_R(s) - iM_R \Gamma_R(s)} \ , \tag{32}$$

and the coefficients β and γ are complex numbers. The energy-dependent widths $\Gamma_R(s)$ are given, as in our approach, by Eq. (28), while the expression for the (real) functions $f_R(s)$ can be found in Ref. [6]. The constants d_R are chosen so that $BW_R^{\mathrm{GS}}(0) = 1$. As stated, this phenomenological parametrization has been used in the fits carried out by the Belle Collaboration [65], allowing a quite successful description of the data throughout the full spectrum. It is represented by the dashed curve in Fig. 2 (in the close-ups in Fig. 3 our curve and the GS curve overlap, and little differences can only be appreciated in the peak region, where our curve shows a slightly better agreement with the data). For comparison we also include in Figs. 2 and 3 the result obtained in Refs. [32] and [33]. The latter corresponds to a dispersive representation of the form factor in the isospin limit, without the inclusion of excited resonant states (we have refitted the parameters according to present Belle data).

4   Low-energy observables

On the basis of the theoretical approach presented in Sect. 2 we can obtain the values of chiral low energy observables. If the expansion of the pion vector form
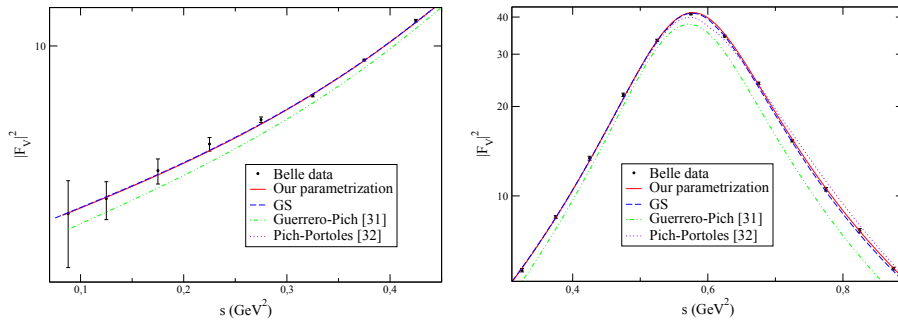
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Figure 3: Two close-ups of Fig. 2 are displayed, corresponding to the low-energy region (left panel) and the peak region (right panel).

factor in powers of s is parametrized as

$$F_V^\pi(s) = 1 + \frac{1}{6} \left\langle r^2 \right\rangle_V^\pi s + c_V^\pi s^2 + d_V^\pi s^3 + \dots ,\qquad(33)$$

from Eq. (17) one has

$$\left\langle r^2 \right\rangle_V^\pi = 6\,\alpha_1 , \quad c_V^\pi = \frac{1}{2}\left(\alpha_2 + \alpha_1^2\right) .\qquad(34)$$

Taking into account the results of our fit (case III, i.e. including isospin-breaking corrections), we obtain

$$\left\langle r^2 \right\rangle_V^\pi = 10.86 \pm 0.14 \text{ GeV}^{-2} , \quad c_V^\pi = 3.84 \pm 0.03 \text{ GeV}^{-4} .\qquad(35)$$

These values are indeed in good agreement with almost all previous determinations made by several authors within various chiral models, see Refs. [28, 29, 33, 78, 79, 80, 81, 82, 83, 84]. In order to go beyond the s^2 term in the expansion, one can make use of the general relation

$$\alpha_k = \frac{k!}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s' \, \frac{\delta_1^1(s')}{s'^{k+1}} ,\qquad(36)$$

which allows to determine the subsequent subtraction constants in the Omnès expression (13). In this way we obtain $\alpha_3 = 29.2 \pm 0.2$ GeV^{-6}, which leads to

$$d_V^\pi = \frac{1}{6}\left(\alpha_3 + 3\alpha_1\alpha_2 + \alpha_1^3\right) = 9.84 \pm 0.05 \text{ GeV}^{-6} .\qquad(37)$$

Previous evaluations for this observable have been carried out in Refs. [78] and [83], leading to 9.70 ± 0.40 GeV^{-6} and 10.18 ± 0.27 GeV^{-6}, respectively. In order to check the consistency of our procedure we have also calculated the constant α_2 from the general relation (36), obtaining $\alpha_2 = (3.7 \pm 0.2)$ GeV^{-4}, in reasonable agreement with the result of the fit. Notice that, even if Eq. (36) is exact, one can expect some deviation from the fitted value of α_2 owing to the ad-hoc treatment of the phase shift above the inelastic threshold in our analysis of the form factor. In the case of α_1 we cannot trust the result from Eq. (36) since the slow convergence of the integral provides a large weight to this high energy contribution.

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Reference	$r_{V_1}^V(M_\rho) \times 10^3$	$r_{V_2}^V(M_\rho) \times 10^4$
VMD [28]	-0.25	2.6
$\mathcal{O}(p^6)$ χPT [28]	$-0.68(26)$	$1.50(44)$
Pich and Portolés [33]	$-0.79(19)$	$1.46(3)$
Our result	$-0.91(16)$	$1.49(1)$

Table 3: Counterterm combinations extracted from $\left\langle r^2 \right\rangle_V^\pi$ and c_V^π in $\mathcal{O}(p^6)$ χPT.

In addition, the observables $\left\langle r^2 \right\rangle_V^\pi$ and c_V^π can be related to two counterterm combinations in the $\mathcal{O}(p^6)$ chiral Lagrangian, namely $r_{V_1}^r(M_\rho)$ and $r_{V_2}^r(M_\rho)$ [28], which are dominated by the vector resonance contributions $r_{V_1}^V(M_\rho)$ and $r_{V_2}^V(M_\rho)$. Our results for these quantities are quoted in Table 3, showing a good agreement with the values previously obtained in Ref. [33] and in the $\mathcal{O}(p^6)$ χPT fit in Ref. [28]. Within VMD these counterterms can be determined by integrating out vector resonances in the framework of a chiral effective theory [19]. Considering just the contribution of the ρ meson resonance, within the Proca formalism one gets [28]

$$r_{V_1}^V = 2\sqrt{2}\frac{F_\pi^2}{M_V^2}f_\chi f_V \ , \quad r_{V_2}^V = \frac{F_\pi^2}{M_V^2}g_V f_V \ , \tag{38}$$

where f_V, g_V and f_χ are effective couplings in the chiral Lagrangian with resonances. Our results for $r_{V_1}^V$ and $r_{V_2}^V$ would lead then to the ratio $f_\chi/g_V = -2.1 \pm 0.5$, far from the phenomenological value $f_\chi/g_V \simeq -0.33$ [28]. This indicates that the role of heavier resonances is crucial in order to describe the $\mathcal{O}(p^6)$ vector driven contributions in χPT, in agreement with Ref. [33].

5 Conclusions

The high quality data on the pion vector form factor obtained at flavour factories demands a correspondingly improved analysis from the theoretical side. In order to describe these data keeping the connection with the underlying strong interaction dynamics, one can take profit of QCD symmetries to reproduce the data in the very low energy domain, and make use of general properties of quantum field theory to extend the analysis to higher energies. In this spirit, we have presented a dispersive representation of the charged pion vector form factor that fulfills the constraints imposed by analyticity and unitarity, and reduces to the result obtained within χPT at low energies.

Our construction is based on the dispersion relation between the form factor and the $\delta_1^1(s)$ phase shift of elastic $\pi\pi$ scattering. The phase shift is obtained from the leading contribution arising in the large-N_C expansion including $\rho(770)$ exchange up the onset of inelasticities, with the further assumption of a smooth growth up to the asymptotic value. In this way we obtain a theoretical expression for the form factor in terms of four parameters, namely M_ρ, F_π, and two subtraction constants α_1 and α_2. The values of these parameters have been determined by performing a fit to the very precise Belle data on the $\tau \to \pi\pi\nu_\tau$ spectral function up to a squared $\pi\pi$ invariant mass $s_{\max} \simeq 1.5$ GeV^2, leading to the results quoted in Table 1. It is seen that the effect of isospin corrections

on the parameters lies below the two percent level. From these results we have determined the pole values of the ρ mass and width and the so-called visible or peak ρ mass. We have also obtained the values of low energy observables and compared the results with those arising from chiral effective theories.

In addition, we have addressed the energy region $s \geq s_{\max} \simeq 1.5$ GeV^2, in which the inclusion of excited states is necessary to get a proper description of $\tau \to \pi\pi\nu_\tau$ data. For this region we have proposed a phenomenological expression for the form factor that takes into account the presence of the resonances ρ' and ρ'', assuming that the effective propagators behave in a similar way as that of the ρ meson. This allows a good fit to the data, leading to values for the ρ' and ρ'' masses and widths similar to those quoted in previous works. It is seen that the curves for the form factor obtained for both energy regions match smoothly at $s \sim s_{\max}$.

As a conclusion, we have seen that the χPT results at low energies supplemented with the leading contributions in the large-N_C expansion are able to provide the input to a dispersive representation of the pion vector form factor which fulfills analyticity and unitarity. On this basis, complemented with a phenomenological description in the high energy region, we have shown that it is possible to reproduce the very precise data on $\tau^- \to \pi^-\pi^0\nu_\tau$ decays throughout all the phase space. This can be used as input to the new hadronic currents of the TAUOLA Monte Carlo generator. On the other hand, our fits lead to the parametrization of the charged pion vector form factor, thus the comparison with a precise determination of the neutral form factor would provide robust information on the $\pi\pi$ contribution to the muon anomalous magnetic moment.

Acknowledgements

Appendix

The explicit form of the loop functions $A_{PQ}(s)$ can be obtained from Ref. [48]. One has

$$A_{PQ}(s) = -\frac{192\,\pi^2\,[s\,M_{PQ}(s) - L_{PQ}(s)]}{s}, \qquad (39)$$

where $M_{PQ}(s)$ and $L_{PQ}(s)$ can be written in terms of new functions Σ_{PQ}, Δ_{PQ}, k_{PQ}, $\bar{J}_{PQ}$ and $\tilde{J}_{PQ}$ as

$$M_{PQ}(s) = \frac{1}{12\,s}(s - 2\,\Sigma_{PQ})\,\bar{J}_{PQ}(s) + \frac{\Delta_{PQ}^2}{3\,s^2}\,\tilde{J}_{PQ}(s) - \frac{1}{6}k_{PQ} + \frac{1}{288\,\pi^2}$$

$$L_{PQ}(s) = \frac{\Delta_{PQ}^2}{4\,s}\,\bar{J}_{PQ}(s) \ . \tag{40}$$

The new functions Σ_{PQ} and Δ_{PQ} are defined by $\Sigma_{PQ} = m_P^2 + m_Q^2$, $\Delta_{PQ} = m_P^2 - m_Q^2$, while k_{PQ} includes the renormalization scale μ:

$$k_{PQ} = \frac{F_\pi^2}{\Delta_{PQ}}(\mu_P - \mu_Q) \ , \tag{41}$$

where

$$\mu_P = \frac{m_P^2}{32\,\pi^2\,F_\pi^2}\log\left(\frac{m_P^2}{\mu^2}\right) \tag{42}$$

(we have taken $\mu = M_\rho$, as in the isospin symmetric case). Finally, the functions $\bar{J}_{PQ}$ and $\tilde{J}_{PQ}$ are given by

$$\tilde{J}_{PQ}(s) = \bar{J}_{PQ}(s) - s\bar{J}'_{PQ}(0)$$

$$\bar{J}_{PQ}(s) = \frac{1}{32\,\pi^2}\left[2 + \left(\frac{\Delta_{PQ}}{s} - \frac{\Sigma_{PQ}}{\Delta_{PQ}}\right)\log\left(\frac{m_Q^2}{m_P^2}\right) - \frac{\nu}{s}\log\left(\frac{(s+\nu)^2 - \Delta_{PQ}^2}{(s-\nu)^2 - \Delta_{PQ}^2}\right)\right] \ , \tag{43}$$

where $\nu = \lambda^{1/2}(s, m_P^2, m_Q^2)$. We note finally that

$$s\,\bar{J}'_{PQ}(0) = \frac{s}{32\pi^2}\left(\frac{\Sigma_{PQ}}{\Delta_{PQ}^2} + 2\frac{M_P^2 M_Q^2}{\Delta_{PQ}^3}\log\frac{M_Q^2}{M_P^2}\right) \ . \tag{44}$$

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